Predictability of the energy cascade in 2D turbulence

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The predictability problem in the inverse energy cascade of two-dimensional turbulence is addressed by means of direct numerical simulations. The growth rate as a function of the error level is determined by means of a finite size extension of the Lyapunov exponent. For error within the inertial range, the linear growth of the error energy, predicted by dimensional argument, is verified with great accuracy. Our numerical findings are in close agreement with the result of TFM closure approximation.

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Unpredictability is an essential property of turbulent flows. Turbulence is characterized by a large number of degrees of freedom interacting with a nonlinear dynamics. Thus turbulence is chaotic (and hence unpredictable), but the standard approach of dynamical system theory is not sufficient to characterize predictability in turbulence.

In fully developed turbulence, the maximum Lyapunov exponent diverges with the Reynolds number thus being very large for typical turbulent flows. Nevertheless, a large value of the Lyapunov exponent do not imply automatically shot time predictability. A familiar example is the atmosphere dynamics: Convective motions in the atmosphere make the small scale features unpredictable after one hour or less, but large scale dynamics can be predicted for several days, as it is demonstrated by weather forecasting. This effect, which can be called "strong chaos with weak butterfly effect" arises in systems possessing many characteristic scales and times. From this point of view, turbulence probably represents the example most extensively studied.

The first attempts to the study of predictability in turbulence dates back to the pioneering work of Lorenz [1] and to Kraichnan and Leith papers [2,3]. On the basis of closure approximations, it was possible to obtain quantitative predictions on the evolution of the error in different turbulent situations, both in two and three dimensions.

A more recent approach to the problem is based on dynamical system theory. One of the first results is the Ruelle prediction on the scaling of the Lyapunov exponent with the Reynolds number [4]. Chaotic properties have been extensively investigated in simplified models of turbulence, called Shell Models, with particular emphasis on the relations with intermittency [5,6]. Because predictability experiments in fully developed turbulence are numerically rather expensive, a similar study on direct numerical simulations of Navier–Stokes equations is still lacking.

In this letter we address the predictability problem for two–dimensional turbulence by means of high resolution direct numerical simulations. Turbulence is generated in the inverse cascade regime where a robust energy cascade is observed [7]. The absence of intermittency corrections makes the problem simpler than in the three–dimensional case: velocity statistics (energy spectrum, structure functions) is found to be in close agreement with self-similar theory à la Kolmogorov.

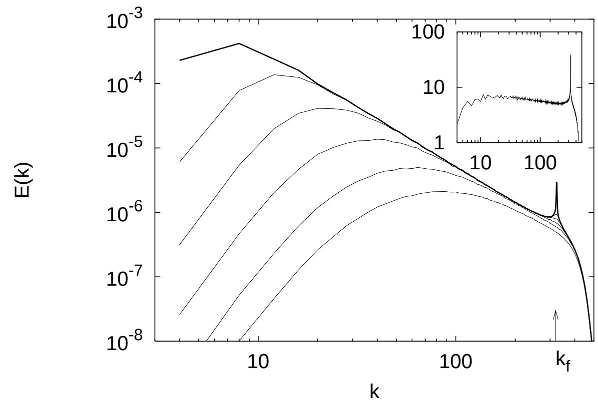

FIG. 1. Stationary energy spectrum $E(k)$ (thick line) and error spectrum $E_\Delta(k,t)$ at time $t = 0.1, 0.2, 0.4, 0.8, 1.6$. $k_f = 320$ is the forcing wavenumber. In the inset we plot the compensated spectrum $\epsilon^{-2/3}k^{5/3}E(k)$.

The model equation is the two–dimensional Navier–Stokes equation written for the scalar vorticity $\omega(\mathbf{r},t) = -\triangle\psi(\mathbf{r},t)$ with generalized dissipation and linear friction

$$\partial_t\omega + J(\omega,\psi) = (-1)^{p+1}\nu_p\triangle^p\omega - \alpha\omega - f \qquad (1)$$

where J represent the Jacobian with the stream function ψ from which the velocity is $\mathbf{v} = (\partial_y\psi, -\partial_x\psi)$. p is the order of the dissipation, $p = 1$ for ordinary dissipation, $p > 1$ for hyperviscosity. As it is customary in numerical simulations, we use hyperviscous dissipation ($p = 8$) in order to extend the inertial range. Although this can affect the small scale features of the vorticity field [9], in our simulations dissipation is not involved in the cascade and has simply the role of removing enstrophy at small scales. The friction term in (1) removes energy at large scales: it is necessary in order to avoid Bose–Einstein condensation on the gravest mode [10] and to obtain a stationary state. Energy is injected into the system by a random forcing δ–correlated in time which is active on a shell of wavenumbers around k_f only. Numerical inte-

gration of (1) is performed by a standard pseudo-spectral code fully dealiased with second–order Adams–Bashforth time stepping on a doubly periodic square domain with resolution $N = 1024$.

Stationary turbulent flow is obtained after a very long simulation starting from a zero vorticity initial field. At stationarity we observe a wide inertial range with a well developed Kolmogorov energy spectrum $E(k) = C\epsilon^{2/3}k^{-5/3}$ (Figure 1). Structure functions in physical space are found in agreement with the self-similar Kolmogorov theory [7].

Starting from the stationary configuration, the predictability experiment integrates two different realizations of the turbulent field and looks at the evolution of the difference (or error) field defined for the velocity coordinates as

$$\delta\mathbf{u}(\mathbf{r},t) = \frac{1}{\sqrt{2}}\left(\mathbf{u}_1(\mathbf{r},t) - \mathbf{u}_2(\mathbf{r},t)\right) \tag{2}$$

From (2) one defines the error energy and the error energy spectrum as [3,8]

$$E_\Delta(t) = \int_0^\infty E_\Delta(k,t)\,dk = \frac{1}{2}\int |\delta\mathbf{u}(\mathbf{r},t)|^2 d^2r \tag{3}$$

Normalization in (2) ensures that $E_\Delta(k,t) \to E(k)$ for uncorrelated fields (i.e. for $t \to \infty$).

Assuming infinitesimal initial errors, the magnitude of the difference field starts to grow exponentially and $E_\Delta(t) \simeq E_\Delta(0)exp(2\lambda t)$ where λ is the maximum Lyapunov exponent of the system. The error growth is this stage is confined at the faster scales in the inertial range, corresponding in our model to the scales close to the forcing wavenumber k_f (see Figure 1). For finite errors, when $E_\Delta(k_f,t)$ becomes comparable with $E(k_f)$, the exponential growth terminate and an algebraic regime sets in. The dimensional prediction proposed by Lorenz [1] assumes that the time it takes for the error to induce a complete uncertainty at wavenumber k is proportional to the characteristic time at that scale, $t \simeq \tau(k)$. Within the Kolmogorov framework, $\tau(k) \simeq \epsilon^{-1/3}k^{-2/3}$. At larger scales the error is still very small in comparison with the typical energy. We thus can write

$$E_\Delta(k', t = \tau(k)) = \begin{cases} 0 & \text{if } k' < k \\ E(k) & \text{if } k' > k \end{cases} \tag{4}$$

By inserting (4) in (3), using the Kolmogorov spectrum for $E(k)$ and inverting the dimensional expression for $\tau(k)$ one ends with the prediction [1,11]

$$E_\Delta(t) = G\epsilon t \tag{5}$$

The numerical constant G in (5) can be obtained only by repeating the argument more formally within a closure framework [2,3,12]. In Figure 2 we plot the time evolution of the error energy $\langle E_\Delta(t)\rangle$ obtained from direct numerical simulations averaged over 20 realizations. Both the exponential regime a small time and the linear regime (5) are visible.

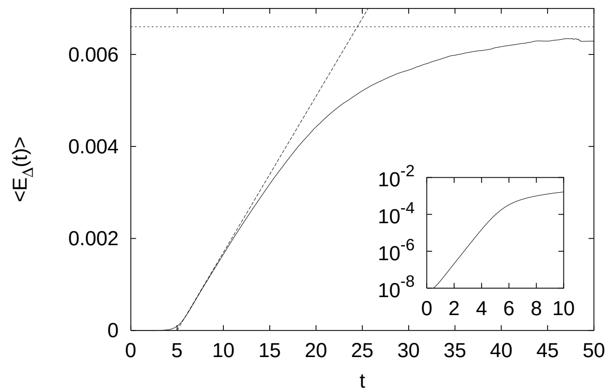

FIG. 2. Average error energy $\langle E_\Delta(t)\rangle$ growth. Dashed line represents closure prediction (5), dotted line is the saturation value E. The initial exponential growth is emphasized by the lin-log plot in the inset.

The dimensional predictability argument given above can be rephrased in a language more close to dynamical systems by introducing the Finite Size Lyapunov Exponent analysis. FSLE is a generalization of the Lyapunov exponent to finite size errors, which was recently proposed for the analysis of systems with many characteristic scales [13–15]. In a nutshell one computes the "error doubling time" $T_r(\delta)$, i.e. the time it takes for an error of size $\delta = |\delta\mathbf{u}|$ to grow of a factor r (for $r = 2$ we have actually a doubling time). The FSLE is defined in term of the average doubling time as

$$\lambda(\delta) = \frac{1}{\langle T_r(\delta)\rangle}\ln r \tag{6}$$

It is easy to show that definition (6) reduces to the standard Lyapunov exponent λ in the infinitesimal error limit $\delta \to 0$ [14]. For finite error, the FSLE measures the effective error growth rate at error size δ. Let us remark that taking averages at fixed time, as in (5) is not the same of averaging at fixed error size, as in (6). This is particularly true in the case of intermittent systems, in which strong fluctuations of the error in different realizations can hide scaling laws like (5) [16]. From a numerical point of view, the computation of $\lambda(\delta)$ is not more expensive than the computation of the Lyapunov exponent with a standard algorithm.

The same dimensional argument leading to (5), repeated for $T_r(\delta)$, gives the prediction for the FSLE in energy cascade inertial range

$$\lambda(\delta) = A\epsilon\delta^{-2} \tag{7}$$

where the constant A is again not determined by dimensional arguments.

The scaling (7), which can be shown to be not affected by possible intermittency corrections (as in 3D turbulence [13]), is valid within the inertial range $u(k_f) < \delta <$

U where $u(k_f)$ represents the typical velocity fluctuation at forcing wavenumber and $U \simeq \sqrt{2E}$ is the large scale velocity.

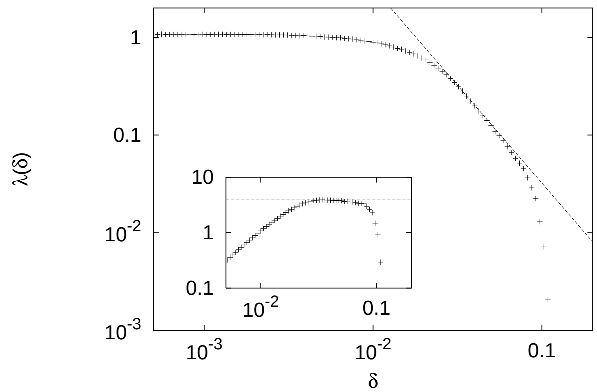

FIG. 3. Finite size Lyapunov exponent $\lambda(\delta)$ as a function of velocity uncertainty δ. The asymptotic constant value for $\delta \to 0$ is the maximum Lyapunov exponent of the turbulent flow. Dashed line represent prediction (7). In the inset we show in the compensated plot $\lambda(\delta)\delta^2/\epsilon$. The line represent the fit to the constant $A \simeq 3.9$.

Figure 3 shows the FSLE computed from our simulations. For small errors $\delta < u(k_f)$ (corresponding to an error spectrum $E_\Delta(k_f, t) << E(k_f)$) we observe the convergence of $\lambda(\delta)$ to the leading Lyapunov exponent. Its value is essentially the inverse of the smallest characteristic time in the system and represents the growth rate of the most unstable features. At larger $\delta > 10^{-2}$ we clearly see the transition to the inertial range scaling (7). At further large $\delta \simeq U \simeq 0.1$, $\lambda(\delta)$ falls down to zero in correspondence of error saturation.

In order to emphasize scaling (7), in Figure 3 we also show the compensation of $\lambda(\delta)$ with $\epsilon\delta^{-2}$. Prediction (7) is verified with very high accuracy which allows to determine the value of $A \simeq 3.9 \pm 0.1$. The constant A relates the energy flux in the cascade to the rate of error growth. In absence of intermittency and with $r \simeq 1$ it is possible to relate (5) to (7). In the present case ($r \simeq 1.12$) one obtains $G \simeq 4.1$. It is interesting to observe that the numerical result is very close to the old prediction obtained by a Test Field Model closure [3] which gives $G = 4.19$. At least from the point of view of predictability, two-dimensional turbulence seems to be very well captured by low-order closure scheme. As a consequence we can exclude, on the basis of our numerical findings, the existence of intermittency effects in the inverse cascade of error. This is a result which is probably of more general interest than the specific problem discussed in this letter.

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[1] E.N. Lorenz, "The predictability of a flow which possesses many scales of motion," Tellus **21**, 289 (1969).
[2] C.E. Leith, "Atmospheric Predictability and Two-Dimensional Turbulence," J. Atmos. Sciences **28**, 145 (1971).
[3] C.E. Leith and R.H. Kraichnan, "Predictability of Turbulent Flows," J. Atmos. Sciences **29**, 1041 (1972).
[4] D. Ruelle, "Microscopic fluctuations and turbulence," Phys. Lett. A **72**, 81 (1979).
[5] K. Ohkitani and M. Yamada, "Temporal Intermittency in the Energy Cascade Process and Local Lyapunov Analysis in Fully-Developed Model Turbulence," Prog. Theor. Phys. **81**, 329 (1989).
[6] M.H. Jensen, G. Paladin and A. Vulpiani, "Intermittency in a cascade model for three-dimensional turbulence," Phys. Rev. A **43**, 798 (1991).
[7] G. Boffetta, A. Celani and M. Vergassola "Inverse energy cascade in two-dimensional turbulence: Deviations from Gaussian behavior," Phys. Rev. E **61**, R29 (2000)
[8] M. Lesieur, *Turbulence in Fluids* 3th edition, Kluver, Dordrecht (1997).
[9] V. Borue, "Spectral Exponents of Enstrophy Cascade in Stationary Two-Dimensional Homogeneous Turbulence," Phys. Rev. Lett. **71**, 3967 (1993).
[10] L.M. Smith and V. Yakhot, "Bose Condensation and Small-Scale Structure Generation in a Random Force Driven 2D Turbulence," Phys. Rev. Lett. **71**, 352 (1993).
[11] T. Bohr, M. Jensen, G. Paladin and A. Vulpiani, *Dynamical Systems Approach to Turbulence*, Cambridge University Press, Cambridge, UK (1998).
[12] O. Metais and M. Lesieur, "Statistical Predictability of Decaying Turbulence," J. Atmos. Sciences **43**, 857 (1986).
[13] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin and A. Vulpiani, "Growth of non-infinitesimal perturbations in turbulence," Phys. Rev. Lett. **77**, 1262 (1996).
[14] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin and A. Vulpiani, "Predictability in the large: an extension of the concept of Lyapunov exponent," J. of Phys. A **30**, 1 (1997).
[15] G. Boffetta, P. Giuliani, G. Paladin and A. Vulpiani, "An extension of the Lyapunov analysis for the predictability problem," J. Atmos. Sciences **55**, 3409 (1998).
[16] V. Artale, G. Boffetta, A. Celani, M. Cencini and A. Vulpiani, "Dispersion of passive tracers in closed basins: beyond the diffusion coefficient," Phys. Fluids **9**, 3162 (1997).